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QUANTIZATION OF GAUGE THEORIES WITH GLOBAL ANOMALIES

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Abstract: Global anomalies, which obstruct the quantization of certain gauge theories in the temporal gauge, get bypassed in canonical quantization.

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I. Introduction:

I must start by saying how privileged I feel in being able to talk at this meeting meant to honour Professor Haridas Banerjee on the advent of his sixtieth birthday. The subject which I have chosen for this occasion was close to his heart a few years back: global anomalies. He pointed out flaws in several claims about the existence of a global anomaly in the SU(2) gauge theory of a Weyl fermion doublet. I will not have anything to say on that issue, but I will argue that even if a global anomaly is present in a gauge theory, it is essentially harmless. I hope this is a result which he will like!

Anomalies of two different types can be involved in the quantization of gauge theories. The existence of divergence anomalies has been known for a long time [1]: certain classical theories have symmetry currents which cease to be conserved after quantization. In case the current is associated with a symmetry which is gauged, there appears to be a problem in the quantization of the theory because the equations of motion of the gauge fields require the current to be conserved. Fortunately, these apparently contradictory features - nonconservation due to the anomaly and conservation due to the gauging - can be ironed out because the anomaly itself can be made to vanish by going to a submanifold of the classical phase space before quantization. Of course, there is a difference from theories with nonanomalous gauge currents. In those theories, there is gauge freedom, which means that the theories can be described in any of an infinite variety of gauges. This is not possible in a straightforward manner in anomalous gauge theories, where the gauge is, as it were, fixed by the anomaly.

A second kind of anomaly - the so-called global, as opposed to the more common local, kind - was discovered in the early eighties [2]. Here the gauge current is conserved, but the group of time-independent gauge transformations is not simply connected. This has serious consequences for Dirac quantization in the so-called temporal gauge. One obtains a representation of the Lie algebra of the group of time-independent gauge transformations in the Hilbert space of states, but this provides in general only projective (multiple-valued) representations of the group itself. When the fermion content is such that the representation is not a true one, there is no state in the Hilbert space which is invariant under the group, so that the subspace of states obeying Gauss's law is trivial. Fortunately, this problem can be avoided by fully fixing the gauge. The difference between theories with global anomalies and anomaly-free theories is very slight, as we shall see.

If one follows the canonical procedure of quantization, it is easy to see that there is no conceptual difficulty in quantizing these theories. So we shall first present this line of argument. But most high energy theorists nowadays think in terms of functional integrals, so we shall also explain the difference between anomaly-free theories and those with anomalies in the context of functional integrals.

II. Canonical approach

Observe that the argument given above (impossibility of imposing Gauss's law) is specific to Dirac's method of quantization, where quantization is done prior to the removal of gauge degrees of freedom, and is to be contrasted with canonical quantization [3], where all constraints and gauge conditions are imposed at the classical level and quantization is carried out on the nonsingular theory. The imposition of Gauss's law and the gauge condition reduces the phase space. The dynamical system that remains can be quantized as usual. As Gauss's law becomes an operator equation in the Hilbert space, this space does *not* carry any nontrivial representation of either the gauge group or its Lie algebra, so that there is no question of any complication involving projective representations in canonical quantization. The enforcement of Gauss's law in this approach may seem to be done by brute force when compared to Dirac quantization, but the point is that it works [4].

III. Functional integral approach

We pass on to the functional integral formulation of the theories. The full partition function of a gauge theory with fermions will be written as

$$Z = \int \mathcal{D}A Z[A], \tag{1}$$

where $Z[A]$ is the exponential of the negatived effective action, obtained by functionally integrating the exponential of the negatived classical action over the fermion fields.

In an anomaly-free theory, $Z[A]$ is gauge invariant. The presence of an anomaly makes $Z[A]$ vary with gauge transformations of A:

$$Z[A^g] = e^{i\alpha(A,g^{-1})} Z[A], \tag{2}$$

where α is an integral representation of the anomaly [5]. The case of a global anomaly involves a special form of α. One way of characterizing a theory with a global anomaly is to say that the full group of time-dependent gauge transformations is disconnected. Thus there is a possibility of distinguishing between transformations not connected to the identity and ones obtainable from the identity by a sequence of infinitesimal transformations. It is only under the former, i.e., the large gauge transformations, that $Z[A]$ does not stay invariant in these theories. To be precise, the transformation is given by

$$Z[A^g] = e^{i\gamma(g)} Z[A], \tag{3}$$

where $\gamma(g)$ cannot be taken to vanish *except* for gauge transformations g connected to the identity.

In anomaly-free theories, the full partition function factorizes into the volume of the gauge group and a gauge-fixed partition function:

$$\begin{aligned} Z &= \int \mathcal{D}A Z[A] \\ &= \int \mathcal{D}A Z[A] \int \mathcal{D}g \delta(f(A^g)) \Delta_f(A) \\ &= \int \mathcal{D}g \int \mathcal{D}A Z[A^{g^{-1}}] \delta(f(A)) \Delta_f(A) \\ &= \int \mathcal{D}g \int \mathcal{D}A Z[A] \delta(f(A)) \Delta_f(A) \\ &= (\int \mathcal{D}g) Z_f. \end{aligned} \tag{4}$$

Here standard Faddeev-Popov notation has been used, with $\delta(f)$ implementing a gauge-fixing condition and Δ_f the corresponding Faddeev-Popov determinant. In deriving the fourth equality, the invariance of $Z[A]$ under a gauge transformation has been used.

The above *decoupling of the gauge degrees of freedom* does not occur in general if an anomaly is present. In this case, one has

$$Z = \int \mathcal{D}g \int \mathcal{D}A e^{i\alpha(A,g)} Z[A]\delta(f(A))\Delta_f(A), \tag{5}$$

in which g and A are seen to be coupled because of the anomaly term α. However, for global anomalies the partition function does factorize:

$$Z = \int \mathcal{D}g e^{-i\gamma(g)} \int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A). \tag{6}$$

One has to be careful here. The phase factors form a representation of the group, so

$$\int \mathcal{D}g e^{-i\gamma(g)} = \int \mathcal{D}(gh) e^{-i\gamma(gh)} = e^{-i\gamma(h)} \int \mathcal{D}g e^{-i\gamma(g)}, \tag{7}$$

where a fixed element h of the gauge group has been used. If it is not connected to the identity, the left and right hand sides seem to differ by a phase factor, indicating that $\int \mathcal{D}g e^{-i\gamma(g)}$ must vanish. This implies that the partition function Z vanishes. In fact, this was given as one of the arguments against the definability of such theories [2]. However, one is really interested in the expectation values of gauge invariant operators:

$$\frac{\int \mathcal{D}A Z[A]\mathcal{O}}{\int \mathcal{D}A Z[A]} = \frac{\int \mathcal{D}g e^{-i\gamma(g)} \int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A)\mathcal{O}}{\int \mathcal{D}g e^{-i\gamma(g)} \int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A)}. \tag{8}$$

The right hand side is of the form $0/0$ because of the presence of the factor $\int \mathcal{D}g e^{-i\gamma(g)}$ in the numerator and the denominator. Although it is formally meaningless, one can hope to interpret this ratio in a sensible way by removing this common vanishing factor. One thus expects

$$< \mathcal{O} >= \frac{\int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A)\mathcal{O}}{\int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A)}. \tag{9}$$

Now (9) is precisely what one gets in the *canonical* approach to quantization. We have considered above the *Lagrangian* functional integral: the singular nature of the Lagrangian has been ignored and all degrees of freedom, physical or unphysical, integrated over. In the canonical approach, on the other hand, the gauge degrees of freedom are removed by fixing the gauge at the classical level [3] and only the physical part of the theory quantized. The functional integration is then over only the physical fields. There are both ordinary fields and conjugate momenta, but the latter are easily integrated over, resulting in functional integrals leading to (9). This is achieved *without* making use of the full partition function which was used in the Lagrangian approach and caused all the problem in this case by happening to vanish.

This simple resolution of the problem does not mean that there is no trace whatsoever of the global anomaly. An interesting consequence of the disconnectedness of the gauge group is that gauge-fixing functions f can be classified. Two functions f and f' belong to the same class if one can find a gauge transformation connected to the identity to go from a gauge field configuration satisfying one gauge condition to one satisfying the other. In this situation, Z_f and $Z_{f'}$ are equal. In general, however, the transformation that is needed will not be connected to the identity. To see what happens in this situation, we can go through the argument which is used, in anomaly-free theories, to show that the gauge-fixed partition function is the same for different gauge functions. Thus,

$$
\begin{aligned}
Z_f &= \int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A) \\
&= \int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A) \int \mathcal{D}g\delta(f'(A^g))\Delta_{f'}(A) \\
&= \int \mathcal{D}g \int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A)\delta(f'(A^g))\Delta_{f'}(A) \\
&= \int \mathcal{D}g \int \mathcal{D}A Z[A^{g^{-1}}]\delta(f(A^{g^{-1}}))\Delta_f(A)\delta(f'(A))\Delta_{f'}(A) \\
&= \int \mathcal{D}A Z[A]\delta(f'(A))\Delta_{f'}(A) \int \mathcal{D}g e^{-i\gamma(g)}\delta(f(A^{g^{-1}}))\Delta_f(A) \quad (10)
\end{aligned}
$$

Were it not for the phase factor $e^{-i\gamma(g)}$, the last integral would be the identity and the right hand side would reduce to the gauge-fixed partition function for the gauge function f'. The two integrals appear to be coupled here. But that is not really the case. Although different gauge field configurations have to be integrated over, only those are relevant for which both $f'(A)$ and $f(A^{g^{-1}})$ vanish, and the second condition picks out one g for each A satisfying the first condition. As A changes continuously – the spacetime is taken to be compactified – g varies in a fixed homotopy class, so that $\gamma(g)$, which depends only on the class, remains unchanged. Consequently, the factor can be pulled out and one can write

$$Z_f = e^{-i\gamma(g_0)} Z_{f'}, \tag{11}$$

where g_0 is an element of the relevant homotopy class, which is determined by the gauge functions f and f'. It is through these factors that theories with global anomalies differ from anomaly-free theories. But these factors occur only in the partition functions and clearly cancel out in the expectation values of gauge invariant operators, so that *Green functions of gauge invariant operators are fully gauge independent.*

IV. Concluding Remarks

There is one assumption here which has to be pointed out: the possibility of choosing a gauge condition in these theories. Now there is a general theorem [6] asserting that gauges *cannot* be chosen in a smooth way. But it is also known [6] that for the construction of functional integrals, it is sufficient to have piecewise smooth gauges. It should be emphasized that this is supposed to be required even for theories *without* disconnected gauge groups.

To sum up, the problems with gauge theories suffering from global anomalies can be avoided by canonical quantization, where the singular nature of the gauge field Lagrangian is recognized and the constraints properly imposed. Even the Lagrangian functional integral approach can be used if factors of 0/0 are interpreted in the natural way.

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